Stochastic Resonance Phenomena in Quantum Many-Body Systems

Susana F. Huelga[1] and Martin B. Plenio[2]
[1]*Quantum Physics Group, STRI, School of Physics, Astronomy and Mathematics, University of Hertfordshire, Hatfield, Herts AL10 9AB, UK*
[2]*QOLS, Blackett Laboratory and Institute for Mathematical Sciences, 53 Prince's Gate, Exhibition Road, Imperial College London, London SW7 2PG, UK*
(Dated: February 1, 2008)

We discuss stochastic resonance (SR) effects in driven coupled quantum systems. We construct dynamical and information theoretic measures of the system's response that exhibit a non-monotonic behaviour as a function of the noise strength. We analyze the relation between lack of monotonicity in the response and the presence of quantum correlations and identify parameter regimes where the breakdown of a linear response can be linked to the presence of entanglement. We also show that a chain of coupled spin systems can exhibit an array-enhanced response, where the sensitivity of a single resonator is enhanced as a result of nearest-neighbour coupling. These results enlarge the domain where SR effects exist and should be observable in arrays of superconducting qubits.

PACS numbers: PACS numbers: 03.67.-a, 03.67.Hk

Recently, there has been an increasing interest in the observation of entanglement and correlation phenomena in arrays of super-conducting and other solid state qubits realizations [1]. In general, the presence of noise and finite temperatures is considered detrimental and noise levels are thus aimed to be minimized. Here, however, we will explore situations where it is advantageous to maintain a finite, not necessarily minimal, noise strength.

The response of an open quantum system to a weak periodic forcing can exhibit a resonance-like dependence on the noise strength [2]. A canonical example to illustrate this form of *quantum SR (QSR)* is provided by the periodically driven biased spin-boson model [3]. Despite quantum coherence was believed to contribute to the disappearance of SR effects [4], recent work has shown that QSR should also be displayed by systems whose dissipative dynamics obeys conventional Bloch equations [5] and different experimental realizations in quantum optics have been proposed [6]. We will show that noise-enhanced effects are also present in the steady state response of quantum-mechanically correlated systems and quantify the presence of stochastic resonance in terms of both dynamical and information theoretic measures.

Our system consists of an array of N driven, longitudinally coupled spin-1/2 systems (qubits). The system is subject to a noisy environment modelled by a set of harmonic oscillators such that each qubit couples transversely to its own bath. The global Hamiltonian is given by,

$$\begin{aligned} H \;=\; & -\sum_{i=1}^{N}\frac{\omega_0^i}{2}\,\sigma_z^i+\sum_{k,i}\omega_k^i(a_k^i)^{\dagger}a_k^i+\sum_{i=1}^{N}\sigma_x^iX^i \\ & -\sum_{i=1}^{N-1}J\,\sigma_z^i\otimes\sigma_z^{i+1}+\sum_{i=1}^{N}\Omega_i\left(\sigma_+^ie^{-i\omega_L^it}+h.c.\right), \end{aligned}$$

where $\hbar=1$, $X^i=\sum_k C_k(a_k^i+{a_k^i}^{\dagger})$ denotes the bath's *force operator* and $\sigma_+^i=|1\rangle_i\langle 0|$. The external driving is parameterized by its intensity, as given by the Rabi frequency Ω_i, and the detuning from the qubit frequency $\delta_i=\omega_0^i-\omega_L^i$ [7]. We will consider situations where the driving is weak and the external Rabi frequency is smaller than the interqubit coupling, $\Omega<J$ [8]. Within the rotating wave approximation and for a Markovian bath, we obtain an effective Hamiltonian for the N-qubit array,

$$H_{\rm eff}=H_{\rm coh}-i\sum_{i=1}^{N}\Gamma_i(\bar n+1)\,\sigma_+^i\sigma_-^i-i\sum_{i=1}^{N}\Gamma_i\,\bar n\,\sigma_-^i\sigma_+^i,$$

where, $H_{\rm coh}=-\sum_{i=1}^{N}\frac{\delta_i}{2}\,\sigma_z^i-\sum_{i=1}^{N-1}J\,\sigma_z^i\otimes\sigma_z^{i+1}+\sum_{i=1}^{N}\Omega_i\,\sigma_x^i$ is the coherent part of the Hamiltonian in the interaction picture. This master equation treatment is valid in the parameter regime $\Omega_i/\omega\ll 1,\Gamma_i\bar n/\omega\ll 1,\delta_i/\omega\ll 1$ and $J/\omega\ll 1$, where $\omega=\min\{\omega_0^i,\omega_c\}$ for a suitable frequency cut off ω_c of the bath, and all expression in this work will then be correct to lowest non-trivial order in $\Omega_i,\Gamma_i,\delta_i$ and J [9].

The noise strength on qubit i at a temperature T is given by the product $\Gamma_i\,\bar n$, where the explicit functional form of the decay rate Γ_i depends on the spectral properties of the bath and $\bar n$ denotes an effective *boson* number that depends on the bath's temperature T; both parameters are, in principle, controllable. For instance, for superconducting qubits, the parameter Γ may be a measure of the fluctuations in the gate voltages which, if desired, may be amplified on demand [10]. As it will be clear below, our argument is that the steady state response of the system, as quantified by different figures of merit, will be optimized at intermediate noise levels and therefore, trying to reduce the environmental noise to as small as possible values, does not necessarily provide an optimal universal strategy to maximize coherent effects. Let us consider first the case where $N=2$ and $T=0$. By integrating the time evolution $\dot\rho=-i[H_{\rm eff},\rho]+\sum_i\Gamma_i(\bar n+1)\,\sigma_-^i\rho\sigma_+^i+\sum_i\Gamma_i\bar n\,\sigma_+^i\rho\sigma_-^i$,

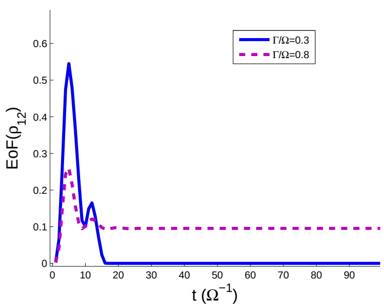

FIG. 1: Entanglement time evolution for two weakly driven qubits with longitudinal coupling of strength $J/\Omega = 1.5$ at $T = 0$. If the noise strength Γ is sufficiently large, the system is inseparable in the steady state (pink dashed line).

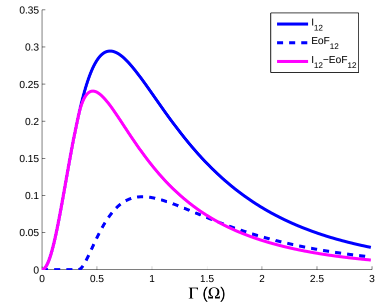

FIG. 2: Stochastic resonance phenomena quantified in terms of information theoretic measures for a systems of two coupled and weakly driven spins ($J/\Omega = 1.5$) at a zero temperature. Presented results are for $\delta = 0$ but deviations up to $\delta/\Omega \sim 10^{-2}$ yield very small deviations from the exact resonance behaviour.

for fixed values of the coupling J and the driving $\Omega = \Omega_1 = \Omega_2$, we can analyze whether our system of weakly driven qubits, initially prepared in their ground state, develops quantum correlations in time. We employ the entanglement of formation E_F to quantify bipartite entanglement [11]. In Figure 1 we observe that the system will be entangled in the steady state only for certain finite values of Γ. Perhaps surprisingly, it is the largest value of the noise strength the one that yields steady-state entanglement. The steady state of the system can be computed analytically. Calling $r = \Gamma/\Omega$, $s = J/\Omega$ and $t = r^2 + 1$,

$$\rho_{12}^{ss} = \frac{1}{k}\begin{pmatrix} t^2 + 4r^2s^2 & 2sr^2 + irt & 2sr^2 + irt & 2irs - r^2 \\ \dots & t & r^2 & ir \\ \dots & \dots & t & ir \\ \dots & \dots & \dots & 1 \end{pmatrix}, \tag{1}$$

where $k = 3 + 2r^2 + t^2 + 4r^2s^2$ and $\dots$ refer to the suitable complex conjugate matrix element. The system is entangled, and have a negative partial transpose, only if $\Gamma > \Gamma_{th}$, where

$$\Gamma_{th} = \frac{\Omega^2}{2J}, \tag{2}$$

is the noise threshold. If $\Gamma < \Gamma_{th}$, the state is separable. This behaviour is illustrated in Figure 2 where the dashed line corresponds to the bipartite entanglement in the steady state as quantified by the E_F as a function of the noise strength Γ. As a result of the constraint given by Eq.(2), any entanglement measure exhibits an initial domain of vanishing entanglement for weak noise where the state is separable. The smaller the qubit interaction strength J, the larger the value for the noise required for the driven spins to be entangled. When Γ rises above threshold, the steady state entanglement increases monotonically up to a maximum at certain optimal noise strength and decreases steadily for higher values of Γ [12]. This functional form for the bipartite entanglement in the domain where $\Gamma > \Gamma_{th}$ is reminiscent of stochastic resonance [2] and it had been observed before in the context of incoherently driven quantum systems [13]. However, to argue for the system to display SR in the conventional sense, we need to construct suitable measures of the system's information content and show the characteristic non-monotonic behaviour as a function of the noise strength that is typical of an SR response. A possible information-theoretic measure is provided by the system's mutual information $I_{12} = S_1 + S_2 - S_{12}$, where S denotes the von Neumann entropy, $S(\rho) = -tr(\rho \log_2 \rho)$ [14]. The blue solid line in Figure 2 is the mutual information of the steady state ρ_{12}^{ss}. Correlations increase monotonically up to a maximum corresponding to a certain optimal noise strength above which I_{12} decreases. This characteristic response is also obtained for the difference between the mutual information and the entanglement (pink solid line). Longer chains, as detailed later, also exhibit this type of non-monotonic response and we therefore argue that SR can be observed beyond the purely incoherent regime analyzed in [14].

Alternatively, stochastic resonance can be characterized using a dynamical measure of the system's response to the external driving. In the case of single quantum systems, it has been proposed to use as an appropriate figure of merit the expectation value of a suitable Pauli operator [15]. This suggests considering local observables of the form $\mathcal{S}_N = \langle (1/N) \sum_{i=1}^N \sigma_\xi^i \rangle$ ($\xi = x, z$) to characterize the system's dynamical response in the multipartite scenario. Note that $\mathcal{S}_N$ is easily accessible experimentally and may be even measured without local control. We can evaluate analytically the form of the signal $\mathcal{S}_2 = \langle (\sigma_x^1 + \sigma_x^2)/2 \rangle = 4sr^2/k$, which is non-monotonic as a function of both the coupling J between qubits and the coupling to the reservoir, reaching a maximum at certain intermediate values. The signal $\mathcal{S}_2$ is maximum for $\Gamma = \sqrt{2}\Omega$, independently of J, with $\mathcal{S}_{2,\max} = s/(2 + s^2)$. The amplitude of the signal is maximal for $s = J/\Omega = \sqrt{2}$ and as the ratio s increases, the response becomes weaker

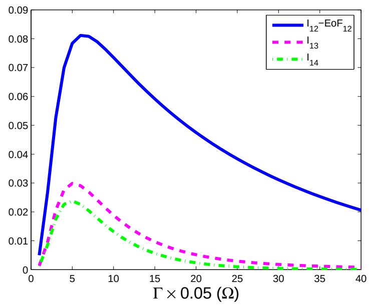

FIG. 3: System's response for a chain of $N = 4$ qubits at $T = 0$ with $J_i/\Omega_i = 1.5$, $(i = 1 \ldots 4)$ as quantified by an information theoretic measure, the quantum mutual information. Under the evolution given by H_{eff}, entanglement is restricted to nearest neighbours only and $E_{F_{13}} = E_{F_{14}} = 0$.

and loses its resonance-like shape, resembling the shape of a saturation curve . In general, for arbitrary values of Ω and J, the transition from linear to non-linear response has no direct relation with the presence of quantum correlations in the system. The system will be separable in the linear region if $\Gamma_{th} > \sqrt{2}\Omega$ while it can be entangled and responding linearly if $\Gamma_{th} < \Gamma < \sqrt{2}\Omega$. However, tuning the nearest neighbour coupling J to the value $\Omega/\sqrt{8}$ yields a maximal response at exactly the transition point from PPT $\mapsto$ NPT states, i.e. from separable to entangled subsystems. As a result, if $\Omega < \sqrt{8}J$, we can ensure that the breakdown of the monotonic response when varying Γ implies the presence of entanglement in the system. This happens in the regime where $\Omega < J$ for both the dynamical response $\mathcal{S}_2$ and the SR information theoretic measures: The maximum values of the response are reached for a noise strength above threshold and the lack of monotonicity allows to conclude that there is entanglement in the system. Note, however, that the opposite conclusion, i.e. monotonicity $\Longrightarrow$ separability, is not correct in general, even for weak driving.

SR phenomena, as quantified by information theoretic and dynamical measures, should also be observable in chains of longitudinally coupled weakly driven spin systems. With the type of interaction we have considered, quantum correlations are confined to nearest neighbours and retain the same qualitative behaviour discussed in the $N = 2$ case, with quantum correlations only arising above a noise threshold that decreases as function of N. In Figure 3 we have characterized the system's response for a chain with $N = 4$ in terms of the quantum mutual information between qubits 1 and j, $(j = 2, 3, 4)$. We observe a resonance-like response as a function of the noise strength at $T = 0$ for both nearest and distant neighbours. So far we have considered the environment to be at zero temperature. All phenomena described so far are robust in the presence of a finite T, with the net result that maximum values in both information theoretic and dynamical measures of SR are reduced with increasing T (See Figure 5 as an illustration for typical superconducting qubits operating temperatures). The mutual information I_{ij} decreases monotonically with $\bar{n}$, as illustrated by the blue solid line in Figure 4 for qubits 1 and 2 in a chain with $N = 4$, and so does the steady state entanglement, which becomes zero for T sufficiently high. We should stress that I_{12} quantifies 'total' correlations, not distinguishing between quantum and classical contributions. Given that I_{ij} measures total correlations, and the E_F is a measure of entanglement(but not the only one), their difference $I_{ij} - E_{Fij}$ may be viewed as a reasonable attempt to measure the classical correlation content of a quantum state. This quantity displays the characteristic SR shape as T increases, with the maximum response being reached at the point where $E_{F_{12}} = 0$ (pink solid line in Figure 4) [16].

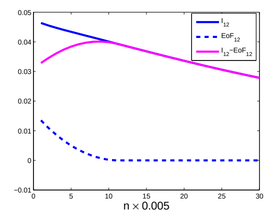

FIG. 4: Information theoretic measures as a function of the bath's mean boson number (temperature) for a chain of N=4 spin with $J_i/\Omega_i = 1.5$ and $\Gamma_i/\Omega_i = 1$, $i = 1 \ldots 4$.

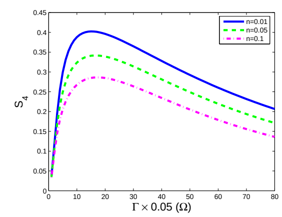

FIG. 5: System's response for a chain of $N = 4$ qubits with $J/\Omega = 1.5$ as quantified by $\mathcal{S}_4 = \langle \sum_{i=1}^{4} \sigma_x^i/4 \rangle$ as a function of the noise strength Γ and for different values of the external temperature.

The non-monotonicity of the response is also apparent in the dynamical measure $\mathcal{S}_4$, plotted in Figure 5 for increasing values of the mean thermal boson number $\bar{n}$. At a given T, the value of Γ that maximizes the response is now a function of both Ω and J and a numerical analysis shows that we can link entanglement and lack of monotonicity whenever outside the regime where $\Omega \gg J$. It is interesting to compare this global response with the individual signal that can be obtained from each qubit alone. Classical stochastic resonators are known to display an array-enhanced SR, where the collective dynamics yields the amplification of some suitably defined signal-to-noise ratio of a single oscillator [17, 18]. On exact resonance, the steady state expectation value $\langle \sigma_x \rangle$ for a driven isolated qubit transversely coupled to a bosonic environment at zero temperature is strictly zero, given that $\langle \sigma_x \rangle = \rho_{01} + \rho_{10} \sim \Omega\delta$. For a finite detuning, the single qubit response is a monotonically decreasing function of the noise strength. Specifically, $|\ \langle \sigma_x \rangle_{ss}\ | = \Omega\,\delta/(\delta^2 + (\Gamma/2)^2 + (\Omega/2)^2)$. Introducing interqubit coupling leads to a build up of a real part in the single qubit coherences which is proportional to the coupling strength J to first order. For $N = 2$,

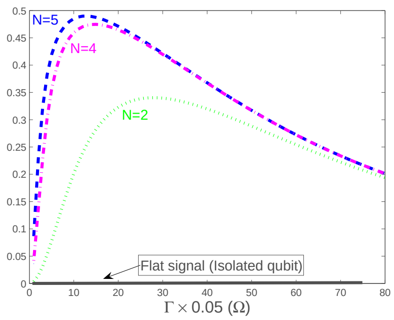

FIG. 6: Single qubit steady state response for weakly driven arrays of 2, 4 and 5 qubits with $J/\Omega = 1.5$ and $\bar{n} = 0.01$. The zero signal corresponds to a resonantly driven uncoupled oscillator. Interqubit coupling yields an enhanced response as a function of the number N of oscillators in the array.

$\langle\sigma_x\rangle = \frac{4J\,\Gamma^2}{k\,\Omega^3}$ and the single qubit response displays the typical SR profile, while increasing the size of the array yields a sharper signal with a maximum value that increases as a function of N and that is obtained for increasingly smaller values of the noise parameter, as noted in Figure 6. The dashed-dotted line corresponds to the signal for the first qubit in an array of $N = 4$, while the dashed line corresponds to the response of the first qubit alone in an array of 5 spins. If we compare the numerical value of the single qubit expectation values with the global response specified in Figure 5, we note that the single qubit response is enhanced as a result of the coupling, which shows the persistence of array enhancement effects in chains of quantum spin systems. It seems clear from the present analysis that stochastic resonance phenomena predicted for coupled classical resonators, including those obeying an Ising model [19], may also be observed within coupled qubits and coexist with the presence quantum correlations. These results are amenable to experimental verification on a variety of proposed qubit realizations [20]. Solid state architectures, and in particular superconducting qubits [10], combine noisy environments with a degree of qubit-environment tunability that made them particularly suited to demonstrate SR in the terms discussed in this Letter.

We are grateful to A. Bulsara, M. Dykman, T. Clark, J. Ralph and D. Tsomokos for useful discussions and to the EPSRC QIP-IRC, the EU Integrated Project QAP, and the Royal Society for support.

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[20] We have examples that demonstrate that our results persist beyond the Born-Markov framework. A detailed analysis of QSR in multipartite systems subject to a non-Markov environment is currently underway and will be presented in a forthcoming paper.